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ABSTRACT

Methodological issues in the use of protocol analysis for research into human problem solving processes are examined through a case study in which two students were videotaped as they worked together to solve mathematical problems "out loud." The students' chosen strategic or executive behavior in examining and solving a problem was studied, focusing on heuristic strategies. A macroscopic framework capturing the essential problem solving elements is described. Protocol episodes include reading, analysis, exploration, planning/implementation, verification and transitions between episodes where executive decisions can take place. The range and interrelationship of variables are discussed which affect the kinds of information emerging from verbal methodologies including the number of persons being taped, the degree of experimenter intervention, the nature and degree of freedom within instructions and intervention, the nature and comfort of the subjects' environment, and task variables. The uses and dangers of protocol analyses in research are discussed. (CM)

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On the Analysis of Two-Person

Problem Solving Protocols

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1. Overview

This paper is one of a pair which, together, try to sketch out some of the issues that should be taken into account when one uses certain "verbal methods" (clinical methodologies or protocol analysis) for research into human problem solving processes. This paper is primarily a case study in one methodology, in which two students are videotaped as they work together to solve mathematical problems "out loud." The focus will be on the advantages, and disadvantages, of this particular methodology -- or more properly, on those aspects of cognitive processes that this methodology will illuminate and those which it will obscure. The context for this discussion is treated at some length in the companion paper, "Beyond the purely cognitive: Metacognition and social cognition as driving forces in intellectual performance." A brief discussion of that context is given in section 3.

2. Background

In recent years there has been a resurgence of the use of verbal data for research into the nature of human cognitive processes. Such research takes as its data the verbal reports produced by individuals or groups of subjects in a variety of circumstances: through retrospection or introspection, in structured or unstructured "clinical" interviews, in "speak aloud" problem solving sessions, with or without experimenter intervention. Through the period of the Gestaltists' major influence, the analysis of verbal reports or introspections was considered methodologically sound, if not the primary source of information regarding complex human cognitions. However, verbal methodologies fell out of favor with the advent of behaviorism and the rise of "scientific" methodologies for the investigation of cognitive phenomena.

The mental constructs posited by the Gestaltists were unneccessary for (or more accurately, antithetical to) the theoretical foundations of the behaviorists (see, for example, Skinner [1974]). In addition the products of introspection were nct replicable or verifiable. Perhaps more importantly, they were not falsifiable. Thus they could not, it seemed, serve as the foundation for a cumulative scientific effort. In consequence the methodologies that gave rise to such unscientific results were supplanted by more "rigorous" methodologies that promised to yield "good science." Verbal methods were déclassé through the 1960's and the 1970's.

For a number of reasons, perspectives on verbal methods have changed in recent years. Perhaps the major cause of the change is the "legitimization" of protocol analysis as a consequence of its role as a major research tool in artificial intelligerce. Such research (see, for example, Newell and Simon [19/2]) demonstrated that one can design successful problem solving programs for computers, based on principles abstracted from the analysis of human problem solving protocols. These computer programs offered, for the first time, incontrovertible empirical "proof" of the efficacy of certain strategies*, and gave credibility to the methodologies that uncovered them. Another major cause was the impact of Piaget's genetic epistemology in general and, in mathematics education, the impact of Krutetskii's work (Krutetskii, 1976). Piaget's work made it clear that c[illegible] l clinical investigations could give rise to replicable results, to falsifiable hypotheses, and to predictions that could be tested experimentally. In short, clinical investigations could indeed lay the foundations for good science. Krutetskii's

*Technically, they offered proof that the machine implementation of such strategies is possible, not that humans actually use the strategies.

work was not "science" in the unbiased, objective sense that we take it. However, it dealt with issues from a perspective that appeared to provide more direct explanations of students' "real" mathematical behavior than the work coming from many "scientific" studies. This caused much interest in his methodologies (and Soviet "teaching experiments" in general.)

Indeed, one cause of the resurgence of verbal methods is the increased sophistication of the research community and its more balanced perspectives on the methodologies that supplanted them. The limitations of the statistical methodologies began to emerge as it became clear, from a lack of clear-cut results in the empirical literature, that there are often (for example) many more variables in "treatment X vs. treatment Y" comparisons than are being controlled for in supposedly "tight" experimental designs. It became clear as well that the difficulties in extrapolating results from well designed laboratory studies to more complex cognitive phenomena, and to more complex environments, had been seriously underestimated. Calls were made (e.g. Kilpatrick, 1975) for the use of clinical investigations to determine, in exploratory fashion, the spectrum of important "mathematical abilities." More recently, the cognitive community has begun to recognize the importance of "other than purely cognitive" influences on what were once taken as "purely cognitive" actions. Thus the role of metacognitions and social cognitions (belief systems, etc.) as driving forces in human intellectual performance is coming to receive more attention (see, e.g., Brown, 1978; D'Andrade, 1981; Lawson, 1980). A range of exploratory methodologies, often verbal, has been developed to deal with such questions. Hence for many different reasons, verbal (clinical or protocol) methods are used with increasing frequency as research tools. Yet,"while increasingly popular, protocol

methods have not yet received thorough methodological analysis. Little is known concerning their fundamental natures, the rationales underlying their use, and their reliability" (Ginsburg, Kossan, Schwartz, and Swanson, in press). Such analyses are beginning to emerge, the Ginsburg et al paper being one of them. Also, Psychological Review has published two recent analyses of the effects of speaking aloud methods. Nisbett and Wilson's [1977] title, "Telling more than we know: Verbal reports on mental processes" suggests its conclusions. Ericsson and Simon [1980] conclude that certain kinds of "talking aloud" instructions -- those that ask for verbalization as one solves a problem, without calling for explanations (elaborations or retrospections) of what one is doing -- do not seem to affect people's performance while solving problems. This paper and its companion will suggest that that conclusion needs to be further qualified. Some of the relevant issues and variables are characterized next.

3. Context

Issues regarding the validity and generality of verbal methods are singularly complex and subtle. Any particular framework for gathering and analyzing verbal data will illuminate certain aspects of cognitive processes and obscure others.* Perhaps more importantly, it may appear to illuminate many behaviors that have, in actuality, been distorted in a number of subtle ways. Each methodology is a lens (or filter, if you will) through which intellectual performance is being viewed. Thus the selection of any particular methodology for investigation may well determine what the experimenter does

*To be accurate, I should talk of "social-cognitive" processes, in the sense that the "cognitions" being studied take place in a social context, which may well determine what the experimenter sees. The discussion below will clarify this point.

or does not see. In turn, this may affect the theoretical constructs that are derived from these observations. Since there is great potential for distortion in this arena, the experimenter wishing a sense of the "whole cognitive picture" should consider using a range of complementary (verbal and other) methodologies, and must be extremely cautious in interpreting the results obtained from a body of methodologically similar studies.

A wide range of variables affect the kinds of information that emerge from verbal methodologies. Some of them are sketched briefly here.

a. The number of persons being taped.

Radically different types of behavior emerge in single-person, two-person, and small group (say three to five people) protocols. The prevailing assumption is that single-person protocols give rise to the "purest" cognitions, uncontaminated by social concerns. However, the task environment itself imposes certain constraints upon the subject(s),** and the discomfiting effects of a task environment may be strongest when a person is solving a problem alone, rather than with the (intellectual and social) support of a peer. Certain behaviors become more prominent, and easier to observe, with more than one subject (e.g. decisionmaking). However, observing other aspects of behavior is made more difficult. One dominant member of a group can skew discussions to the point where they reflect only that person's ideas; solutions may proceed in parallel, or with (positive or negative) reinforcement from the interactions. The more people involved, the more obvious the social dynamics. There are no value judgments attached to these characterizations -- each serves its purpose, and one should simply choose

*These "environmental" constraints lessen with the maturity and training of the subjects. However (see below) college seniors still feel them strongly.

the one(s) suited to the ends that one has in mind. If one is interested in making artificial intelligence models of competent problem solving performance, [e.g. Newell and Simon, 1972] then the most appropriate methodology may well be to perform the detailed analyses of single-person protocols. If one wishes to elucidate certain kinds of decisionmaking behavior, [e.g. Schoenfeld, in press] two-person protocols may be appropriate. If one wishes to make statements about students' "real, social" cognitive behavior [e.g. Lesh's "applied problem solving project" and Noddings' analyses of group interactions] then larger groups are appropriate.

b. The degree of intervention

Verbal methods include a continuum of experimenter obtrusiveness that ranges from near invisibility (covert or non-interventionist observations of people in natural settings) to positions of central importance (experimenters inducing "cognitive dissonance" in clinical interview settings). Each serves certain ends in particular situations. If, for example, an experimenter is interested in determining the "Van Hiele level" of a student on geometry tasks (or the Piagetian level of a subject on a particular task), and exploring corollary behavior on other tasks, then a large degree of intervention is almost mandatory. If, however, the experimenter wishes to see how a student copes with difficult problems (what the student pur ues, whether the student goes off on "wild goose chases," etc.), then interve ion may be inappropriate. Indeed, asking the student "why did you do X?" may have a dramatic effect on the student's behavior. Up to that point, the student may not have considered the question. There is, first, the chance that the answer to the question is "manufactured." Second and equally important, the student is now aware that the experimenter is interested in

how such choices are made. The student may begin to reflect on on those choices while working on the given task, and behave from that point on in a manner very different than he or she would otherwise have behaved.

c. The nature and degrees of freedom in instructions and intervention

The kind of instructions subjects are given has a strong effect on what they produce. For example, asking the subject to reflect upon his or her problem solving processes does have an effect on performance [Ericsson and Simon, 1980]. Yet such reflection may point to behaviors that might otherwise be unseen. In clinical interviews there are tradeoffs between standardization on the one hand and experimenter freedom on the other; one has a certain degree of reliability in the first case, and the potential for probing interesting behaviors in the second.

d. The nature of the environment and how comfortable the subject feels in it

To put it simply, students who feel uncomfortable in a particular environment may uniformly exhibit pathological behavior. That the behavior is pathological may not at all be apparant; that may only appear when the experimental conditions are altered. Further, putting students "at ease" may be completely insufficient. The very fact that one is being taped may be enough to induce atypical behavior (see below). Subjects may avoid dealing with the task in any substantive way, in order to avoid feelings of inadequacy when they (as they see it, inevitably) fail at it. They may create certain kinds of behavior, to make it seem as if they know what they are doing. They may select their behavior to tailor it to (what they believe are) the experimenter's wishes. (In the later category, I have tapes in which students say "We could solve it like this, but obviously he doesn't want that.") Or, students may deal with a problem in unusual ways simply because they are in an obviously artificial setting. (A student in one of

Dick Lesh's videotapes, working on a "real world" problem, misread some given information and assumed that he could earn nearly $150 for mowing one person's lawn once. When he was questioned later, he was asked if that seemed like a reasonable figure. It did not. In fact, the student mowed lawns for extra money and knew the figure was unreasonable. But, "it was a hypothetical question, wasn't it?")

e. Task variables

The range of these is tremendous. Does one provide children with manipulatives, for example? How does this affect performance? For a general discussion of task variables in mathematical problem solving, see Goldin and McClintock [1979].

This brief discussion serves to indicate some of the variables that affect the collection and interpretation of verbal data. It is a bare introduction to an area that needs much greater investigation, but it may serve to set the stage for the following discussions.

4. Executive decisions in problem solving: the issue and methodology

As section 3 indicates, one's choice of methodology should be guided by the goals one has for research. The "problem" I set out to investigate, initially, was to explore some of the reasons for students' lack of success in solving "non-routine" problems at the college level. In addition, I wished to examine students' performance before and after a course in mathematical problem solving, in order to determine some of the effects of instruction. Previous work had provided some tools for the investigation, and some ideas as to what mechanisms might contribute to success (or more accurately, to failure). The general arena was an investigation of Pólya-type heuristics and their contributions to problem solving performance.

Earlier studies had indicated that students could learn to use individual heuristics with some degree of competency [Schoenfeld, 1979] and a battery of tests had been used to examine fluency and competency in the implementation of those heuristics. Thus the intention here was not to investigate such competencies. (If it were, some form of detailed clinical probing would undoubtedly have been an appropriate methodology.) Rather, the intention was to investigate a consistent "difficulty" with regard to heuristics. The literature indicated that, while students did seem able to master individual problem solving strategies, the overall effects on their problem solving performance was not nearly as large as was expected or hoped: the problem solving whole was, somehow, less than the sum of its heuristic parts [Wilson, 1967; Smith, 1973; Lucas and Loomer (in Harvey and Romberg), 1980; Goldberg, 1975]. The questions chosen for investigation were: what will students choose to examine in a problem solving situation (and why)? How will they "follow up" on those choices (pursue them, abandon them etc.)? and What is the effect of such "strategic" or "executive" behavior on their problem solving performance? Observe that these questions can be asked about problems that may or may not be amenable to particular heuristic strategies for solution, solved by students who may or may not have the heuristics at their disposal. This was an exploratory study, in that the data (videotapes) were to serve as a source of hypotheses rather than as a test of them. Some of the choices among the variables given above, and the rationales for them, were as follows.

a. <u>The number of persons being taped</u>

For a variety of reasons, two-person protocols provide the richest data for the purpose described above. First, I have found that single-person

protocols (from students, not faculty) tend to generate unnatural behavior in subtle ways. Protocol 1 (appendix 1) was generated by a single student, a senior mathematics major. It is typical of single-person protocols for this problem ("How many cells are there in an average adult human body?") in that much time and effort is spent approximating parts of the body by geometric solids and computing the volume of those solids. In roughly two dozen two-person protocols, not one pair of students has done the same.* This behavior was induced by the setting: the students felt the need to "produce something mathematical" for the researcher. Many-person protocols ease the pressure on the subjects, for the burden of uncomfortableness is shared among the students.

A second reason for not using single-person protocols in these circumstances is the wish to elucidate the nature of the students' strategic decision-making as they work on the problems. For reasons given below, the sessions had little or no experimenter intervention and the subjects were not instructed to explain what they did as they solved problems. In single person "speak aloud" protocols, what appears is often the "trace" of a solution: one sees the results of decisions but gets little insight into how the decisions were made, what options were considered and rejected, etc. When students work together as a team, discussions between them regarding what they should do next often bring those decisions and the reasons for them "out in the open." (A typical dialogue is "Let's do X." "Why? I don't see what good it'll do." "Look...")

*I collected the single-person protocols first, and had begun to construct various (cognitive) explanations for this poor strategic behavior. Only later, when there were two-person protocols for comparison, did it become apparent that the extensive body-volume computations were caused by the social environment.

These reasons suggested the use of many-person ($n \geq 2$) protocols. There are trade-offs with regard to group size. Larger groups provide more "ideas" to manage, and decision making can be more interesting in these circumstances. Also, social dynamics of groups of 4 or 5 may better ameliorate the uncomfortableness of the experimental environment. Two reasons suggested n = 2 as the most suitable choice. First the decisions one faces when "managing" the ideas generated by a group of people may be very different from the decisions one faces when considering the ideas one or two people have generated. For example, one or two people working alone might go off on a "wild goose chase" and squander their problem solving resources that way. In a larger group, the likelihood of someone saying "why?" to the proposed direction is greater, and the resultant behavior different. Also, one or two students may only generate one or two plausible alternatives; a "committee" may generate more. The "perceived solution space" is different, and the resulting behaviors may not reflect those of individuals working alone or in pairs. Second, the focus of this investigation was largely cogniti . With larger groups the degree of social interactions increases, making it more difficult to tease out the "purely cognitive" aspects of students' behavior. These social factors are still (all too) present in 1- and 2-person protocols, however. We have seen how one person "engages in mathematical behavior" to dissipate the pressure of the task environment. In similar circumstances a pair of students may "defuse" the environment by engaging in small talk around the problem. By refusing to take it seriously, they can justify (what is from their perspective) the inevitable failure by telling themselves that they "never really tried." And of course, two-person social dynamics can be quite strong. In all cases, one must take care that the behavior

labeled as "cognitive" is indeed so.

b. The degree of intervention

It is important to keep interventions to an absolute minimum in this kind of study. The idea was to determine the presence (or absence) of certain kinds of "monitoring" and decision making in students' problem solving, and to trace the effects of their presence (or absence). These effects can only be seen if a solution is allowed to run its course. For example, a student may have a "hunch" or some "intuition" about a plausible solution, and begin to work in that direction. From the experimenter's point of view, it may be clear that this is a "wild goose chase," and it may be tempting to find out what prompted the student to pursue it. However, such an intervention precludes the opportunity to observe the effects of such a wild goose chase. After three minutes the student might come to see that it is fruitless, and go on to do something else. Or, the student might never reconsider, and spend the allotted time involved in irrelevancies. In fact, the latter type of behavior occurs all too frequently. In a large number of tapes, students engaged in an essentially irrelevant computation for nearly twenty minutes (the length of the taping sessions). After they ran out of time they were asked what they would do with the result of the computation if it were given to them...and they were unable to say [Schoenfeld, in press]. I now believe that this lack of monitoring is quite typical (though not always this extreme, obviously) of student behavior, and is one of the major contributing factors in students' problem solving failures. This could only be seen, and verified, by letting the solutions proceed unimpeded.

More importantly, in this particular kind of study, experimenter intervention may have a radical effect on the subject's performance and on the

data that is produced. Recall that one purpose of these experiments was to determine the degree to which students reflect on, and oversee, the way that a solution evolves. Suppose, for example, that a student in the midst of a solution is asked to justify a "wild goose chase" or any other strategic decision. Up to that point the student may not have thought about such issues, or dealt with them casually. After the intervention, he or she knows that the experimenter is interested in such questions. In consequence, the student may begin to manufacture such justifications (to be ready for the next intervention). In doing so, that person's behavior may be completely distorted. There is now a training effect, and all data must be interpreted accordingly.

Again, the preceding comments should be interpreted in the context of the goals for the study, which was exploratory. One of its purposes was to explore students' monitoring and executive behaviors, and document the role that they play in students' problem solving performance. Once that information has been gathered satisfactorily, a shift in methodology may be appropriate. The fianl section will mention a revised methodology I am now pursuing.

c. The nature of instructions and interventions

As the reports in Psychological Review [Nisbett and Wilson, 1977; Ericsson and Simon, 1980] indicate, asking students to talk about their problem solving processes during the process of a solution does have an effect on those processes. For that reason students were instructed not to "explain" for the microphone. Rather, they were to work together as a team; what I wanted to hear would emerge from their discussions. Interventions were limited to comments like "I haven't heard you say much in the past three minutes. Are you still working on the problem?" and to specific

responses to questions the students asked.

d. The environment

The setting was obviously artificial (solving non-routine problems for a mathematics professor can hardly be considered natural behavior) and, despite all efforts to the contrary, somewhat stressful. This point should be emphasized. Protocol 1 (appendix 1) was produced by a senior mathematics major who was on a first-name basis with the experimenter. The student was familiar with the entire process (he had done some taping himself as part of a senior thesis). Yet the unfamiliar problem induced great stress, with a resultant effect on the protocol. Similarly, great efforts were taken in all the two-person protocols to put the students at their ease. They were assured that the research was non-judgmental, shown that the videotape machine focused on the pages they were writing and not on their faces, etc. Even so, it is not at all safe to assume that the students would deal with the same problems in anything like the same manner if they worked on them, for example, in their own rooms without a recording device present. The more awkward the situation -- the more obtrusive the recording equipment, the more "unusual" the problem, etc. -- the more likely the "verbal data" is to be affected.

e. Task variables

The subjects were college students, and treated as such. Problems were worked with paper and pencil only, save for "straightedge and compass" geometry constructions, where they were given the tools for the constructions.

5. A framework for examining the protocols

An extensive description of the framework described below, and of the results obtained with it, is given in my article "Episodes and Executive

Decisions in Mathematical Problem Solving" [in press]. In brief, the idea was to create a macroscopic framework that captured the essential elements in the problem solution.* There is one significant difficulty in implementing this idea, a difficulty that has been the downfall of most extant protocol coding schemes: the most important event in a problem solving session may be one that is conspicuous by its absence! For example, appendix 1 of the "Episodes" paper gives the protocol of a tape that had a 20-minute long "wild goose chase" in which the students tried to calculate the area of a geometric figure. At the end of the tape they were asked how they would use the result if they had it, and they could not say. Had they asked themselves, at the moment they set out to do the calculation, what value it would have, they might have avoided wasting their time. But they did not, and the solution was doomed from that point on. Now observe that, as one might expect, conventional coding schemes record overt behaviors in a problem protocol. While this seems to be perfectly natural, the result is that such frameworks bypass the critical element in the protocol described above: the absence of evaluation at a "make or break" point in a solution. Such systems will not point to the reason that the attempt failed. The general idea is to discuss the impact of the (presence or absence of) assessments and consequent decision making of the solution as a whole.

The idea behind the generation of the system is straightforward. Potential "make or break" points in a solution occur whenever the direction of a solution changes radically (when one approach is abandoned for another), or when new

*A qualitative "test" for capturing the essential elements in a protocol is the following. After being given the analysis (coding) of the protocol, are there "surprises" when one sees the tape for the first time? This particular framework seems to pass the test. The other systems with which I am familiar fail it miserably: it is nearly impossible to get a "feel" for what happened from the string of coding symbols.

information arises that might call for such a radical change. The system is designed to identify those points, to characterize the behavior of the students at those points, and to describe the effect of that behavior on the solution.

A protocol is parsed into major "episodes." An episode represents a body of consistent behavior on the part of the problem solver(s). Episodes have one of six labels attached to them: reading, analysis, exploration, planning/implementation, verification, and transition. Once a protocol has been parsed into episodes, one category of "make or break" points becomes obvious: any transition point between episodes is a potential assessment/decision point. Other "executive" decisions should be made at "new information" points.

Appendix 2 provides the full analysis of a protocol, which is given in appendix 3. This analysis provides an example of how the system works. Most of the commentary is self-explanatory.*

6. Discussion

The framework discussed above has proven itself reliable and, I believe, reasonable informative. It seems to capture much of the "essence" of a problem solving session, without getting lost in details. The macroscopic approach allows one to get a sense of apparent causes of success or failure in a

*Letters preceding comments refer to specific parts of the framework. For example, one asks three questions about any reading episode:

a. Have all of the conditions of the problem been noted? Explicitly or implicitly?
b. Has the goal state been correctly noted? (Again, explicitly or implicitly?)
c. Is there an assessment of the current state of the problem solver's knowledge relative to the problem solving task?

In virtually all cases, the questions that lie behind the comments in appendix 2 are clear. They are omitted to save space.

problem solution, and points out the importance both of monitoring solutions and of "executive" decisions in them. The framework is straightforward to implement (three students, in concert, do most of the coding for me) and reliable (their codings and mine have an intercoder reliability exceeding 85%). The framework is also generalizable: it is not domain-specific, and can be adapted easily to study problem solving behavior in other disciplines. However, some caveats are in order.

First of all, this particular methodology offers only one perspective on the problem solving process. It should be coupled with a variety of others (paper-and-pencil tests, clinical interviews to determine mathematical abilities, different protocol methods and different levels of analysis, etc.) in order to provide a reasonably comprehensive picture of problem solving behavior.

Second, there are any number of dangers inherent in the gathering of protocols. A few of these (for example, pathological behavior induced by uncomfortableness, or bad social dynamics) were mentioned above. It is nearly impossible to control for these, or even to be aware of them in any particular protocol. Thus one must exercise extreme caution in providing "purely cognitive" explanations for behavior.

Third, this was an exploratory methodology and has certain limitations. It was non-interventionist, for example, in order to make the case that assessments and managerial decisions play a critical role in determining the success of problem solving attempts. Once that point has been granted, one may well wish to explore "executive" behavior in more detail. I am now trying a variant of the preceding methodology, as follows. A student is first videotaped in the fashion described above. Then the student watches

the videotape and critiques it, explaining the reasons for his or her behavior. These explanations are probed in clinical fashion. This "mixed" methodology will, I hope, allow for a more subtle elucidation of problem solving processes.

Summary

This paper has discussed some of the subtleties involved in the use of verbal methodologies. It has examined in some detail the aims and rationales of a particular methodology, two-person speak-aloud protocols without experimenter intervention, and discussed a framework for analyzing such protocols. Verbal methodologies, if used with care, can help to shed much light on cognitive processes. It is hoped that this is a step in that direction.

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Appendix 1: A single-person protocol

(Reads problem): Estimate, as accurately as you can, how many cells might be in an average-sized adult human body. What is a reasonable upper estimate? A reasonable lower estimate? How much faith do you have in your figures?

I'll think of some approaches I might take to it.

The first one might be just to go by parts of the body that are fairly distinct and try to figure out...

My first possible approach to the problem might be to lock at them as approximations to geometric shapes and try to figure out the volume of each part of the body. And then make a rough estimation of what I thought the volume of a cell was and then try to figure out how many cells fit in there.

I would say take the arm from the wrist up to shoulder and it's approximately a cylinder and it's, I don't know, about 3 or 4 inches in diameter. So you would have, it's about 2 or 1½ inches in radius, squared, times π and the volume of my arm in square inches. So I've got two arms, so I've got two of those.

And now a leg. A leg...think this might be better...there's a little more variance, so I would say a cone might be more appropriate. And the base of my leg is approximately 6 or 7 inches in diameter so you would have $3½^2 \times \pi$ and the height would be... what is my inseam size, about 32 or 34. So you've got to have a 34, and it's a cone so you've got to multiply it by 1/3.

And now the head is very, very roughly a sphere. And so you've got a sphere of... I don't know how many. I don't know, maybe on the average 6" in diameter. That may be a little small, maybe 7" in diameter. And so quick recognition of the formula was $4/3\ \pi r^3$. So I've got 4/3 of whatever my head is cubed, I've got $3½^3$, and what am I missing now?

Oh, torso...very important. Well a torso is...you could say is approximately like a cylinder except with an oval base. So I could figure out what the area may be around is, and I won't calculate this explicitly. Say my waist is about 34" and I could approximate it across here. And if I worked on it I could figure out what the geometry of it of the volume of that ellipse.

S: Well, make a ballpark estimate. I would like to have a number just out of curiosity.

So I've got an ellipse. This may take a while though because my geometry is bad. I've got an ellipse with a perimeter of about 34, and major axis is something along the lines of 18" and the minor axis is maybe...I don't know...8"... And...Oh, geez...

Yea, it's going to be very messy. So I will dispense with that, and instead make another rough estimate, and rather assume myself to be...well, I'm not going to bother to do this, since it's not very exact, anyway. But I could draw a circle, a little bit smaller than that maybe. Well that circle has got...how much... something between 8 and 18, and looking at this I guess you have to stretch and elongate it in the width more than the height...closer to 18...and say 14 in diameter. So that would mean 7" in radius. So, I've got $\pi \times 49$. And that would be my guess for that and the height would be...I don't know...about 15.

Now, I've covered the torso, the two legs, and the two arms.

Ok, for the hands. I'm going to have to make another rough estimate. If I put my hand into a fist I get a little cylinder of maybe an inch and a half and a height of about 4. So I've got two hands with a height of 4, π and the radius of 3/2.

And I have no idea what I'm going to do with my feet. Well, you could make these into little rectangular prisms. 4 x 2 x 10. No actually that looks about right.

Well, maybe the neck, if we're going to be precise about it is going to be 4" in diameter, so we've got a 2" radius neck. So that would be 4π in area, in volume of it. Yea, $4\pi r^2$. And now I would add all these up. Do I have to add them up too?

S: We'll just call that number capital N, and then I'll get Mr. Knop's calculator and we'll actually do it out of curiosity.

Ok, the number is N. Ok, now that I've got the volume of a body, now I've got to figure out what the volume of a cell might be.

And it seems to me something along the lines (unclear). The diameter of a hydrogen atom is like an angstrom unit, and that's something like ten to the minus ten cm. And that's not going to be anything close to the size of a cell. So, if I had to go with the size of a cell...this is a very rough estimate, it might not even be in the right magnitude...it should be 10,000 to the inch or 10,000 cells to the cm. Maybe I'll make a compromise and say 100,000 cells to the inch is right. So that would give me 10^5. So each one is 10^5 in diameter, so we should call them spheres since that would make it simpler. I would have $10^5/2^2$ times π. Is that right? $10^5/2$...you've got 10^5 to the inch so it would be ten to the negative fifth inches over two for the radius...so square that and multiply by π. So you take that and divide it by π.

And I'm going to say that that should give you the volume, but somehow I'm not convinced that that's the case. Well, maybe it would be right because you're going to have a ten to the minus ten in the denominator there, and you multiply these things are going to come out to a good 1000 or so. So hopefully a couple thousand square inches or so when you multiply it...

The student was told that he had computed the area of a circle rather than the volume of a sphere. He made the correction, and then computed all the volumes with the help of a calculator (to 4 place accuracy before rounding off).

Appendix 2*

The Full Analysis of a Protocol

Appendix 3 presents the full protocol of two students working on the following problem:

> Consider the set of all triangles whose perimeter is a fixed number, P. Of these, which has the largest area? Justify your answer as best you can.

Student K is the same student that appeared in protocol 1. Student D (not the same as student D in protocol 2) was a freshmen with one semester of calculus behind him. This protocol was taken at the end of my problem-solving course, while protocols 1 and 2 were taken at the beginning.

The parsing of protocol 3 is given in Figure 1. The analysis given below follows that parsing.

Insert Figure 1 about here

Episode 1 (Reading, items 1, 2)

a. The conditions were noted, explicitly.

b. The goal state was noted, but somewhat carelessly (items 10, 11).

c. There were no assessments, simply a jump into exploration.

Transition 1 (Null)

a, b, c, d. There were no serious assessments of either current knowledge or of directions to come. These might have been costly, but were not--assessments did come in E_2.

Episode 2 (Exploration, items 3-17)

a. The explorations seemed vaguely goal-driven.

b. The actions seemed unfocused.

*Taken from Schoenfeld (in press).

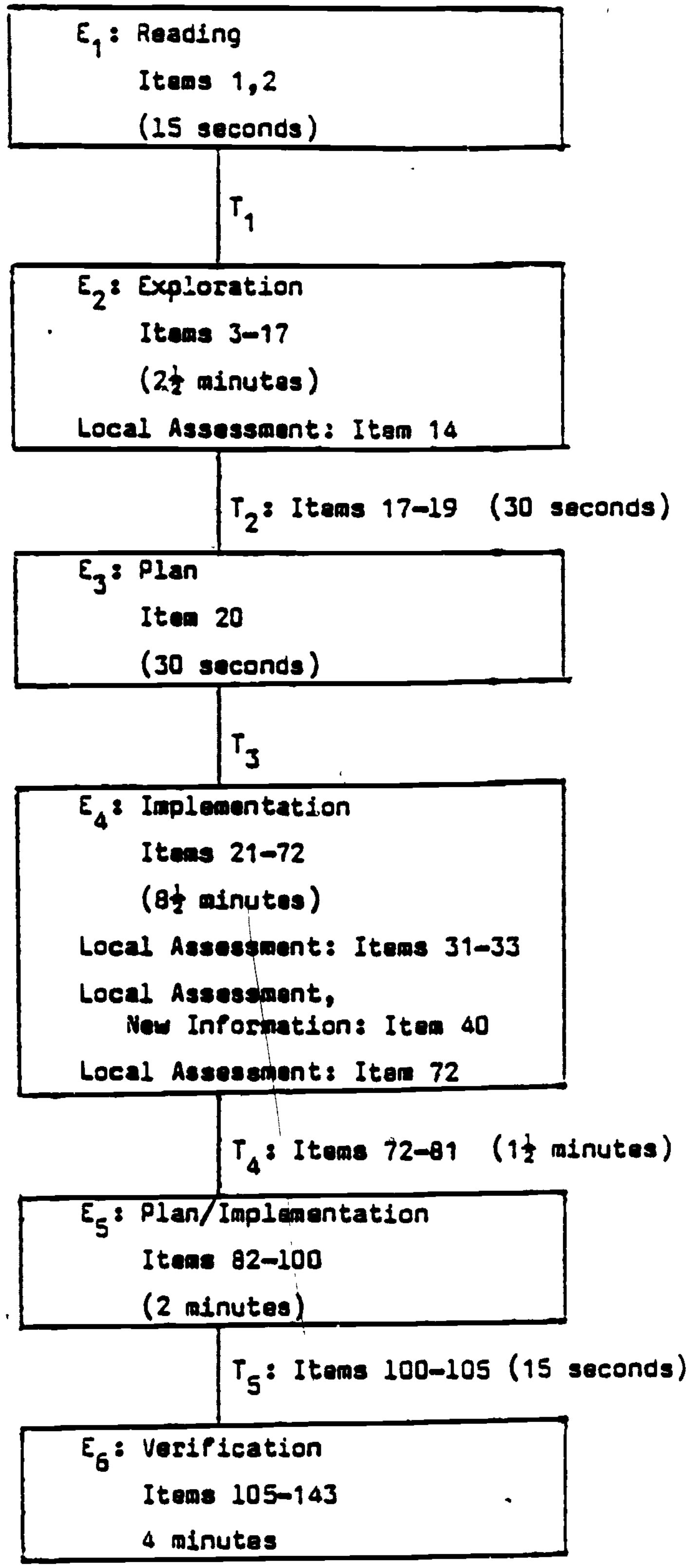


Figure 1

A Parsing of Protocol 3

c, d. There was monitoring, at items 14-17. This grounded the explorations, and led into Transition 2.

Transition 2 (Items 17-19)

a, b, c, d. Assessments were made both of what the students knew, and of the utility of the conjecture they made. The result was the establishment of a major direction: try to prove that the equilateral triangle has the desired property, and of a plan (episode 3). NOTE: If this seems inconsequential, contrast this behavior with the transition T_1 in protocol 1. The lack of assessment there, in virtually identical circumstances, sent the students on a 20 minute wild goose chase!

Episode 3 (Plan, item 20)

a. The plan is overt.

b. It is relevant and well structured. As to appropriateness and assessment, see the discussion of T_3.

Transition 3 (Null)

a, b. There was little of value preceding the plan in item 20; the questions are moot.

c. There was no assessment of the plan; there was immediate implementation.

d. The plan was relevant but only dealt with half of the problem: showing the largest isosceles was the equilateral. The "other half" is to show that the largest triangle must be isosceles, without which this part of the solution is worthless. . .a point realized somewhat in item 72, 8 minutes later. The result was a good deal of wasted effort. The entire solution was not sabotaged, however, because monitoring and feedback mechanisms caused the termination of the implementation episode (see the sequel).

Episode 4 (Implementation, items 21-72)

a. Implementation followed the lines set out in episode 3, albeit in somewhat careless form. The conditions were somewhat muddled as the first differentiation was set up. The next two local assessments corrected for that (better late than never).

Local Assessment (Items 31-33)

1, 2, 3. The physically unrealistic answer caused a closer look at the conditions--but not yet a global reassessment (possibly not called for yet).

Local Assessment, New Information (Item 40)

1, 2, 3. The "new information" here was the realization that one of the problem conditions had been omitted from their implementation ("we don't set any conditions--we're leaving P out of that"). This sent them back to the original plan, without global assessment. The cost: squandered energy until item 72.

Local/Global Assessment (Item 72)

This closes E_4. See T_4.

Transition 4 (Items 72-81)

a, b. The previous episode was abandoned, reasonably. The goal of that episode, "show it's the equilateral," remained. This, too, was reasonable.

c, d. They ease into Episode 5 in item 82. (It's difficult to say how reasonable this is. Had they chosen something that didn't work, it might have been considered meandering. But what they chose did work.)

Episode 5 (Plan/Implementation, items 82-100)

a, b. "Set our base equal to something" is an obviously relevant heuristic.

c. They plunge ahead as usual.

d. The variational argument evolved in a semmingly natural way.

e. There was local assessment (item 95). That led to a rehearsal of the sub-argument (item 96), from which D apparently "saw" the rest of the solution. Further (item 100), D assesses the quality of the solution and his confidence in the result.

Transition 5 (Items 100-105)

a, b, c, d. The sequel is most likely the result of a two-person dialectic. It appears that D was content with his solution (perhaps prematurely), although his clarity in explaining his argument in E_6 suggests he may have been justified.

Episode 6 (Verification, items 105-143)

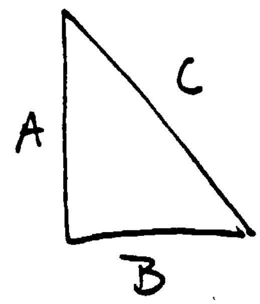
This is not a verification episode in the usual sense. K's unwillingness to rest until he understands forced D into a full rehearsal of the argument and a detailed explanation, the result being that they are both content with the (correct) solution.

Appendix 3

Protocol 3

1\. K: (Reads problem.) Consider the set of all triangles whose perimeter is a fixed number, P. Of these, which has the largest area? Justify your assertion as best you can. All right now what do we do?

2\. D: We got a triangle--well we know we label sides A, B and C.



3\. K: Right. I'll make it a right triangle--all right--A,B, C and the relationship such as that 1/2AB = Area and A+B+C = P and $A^2 + B^2 = C^2$ and somehow you've got an area of one of these in the perimeter.

$\frac{1}{2}AB = A$

$A+B+C=P$

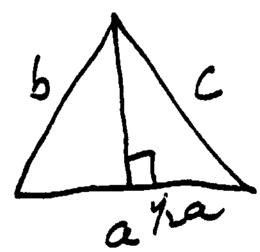
$A^2+B^2=C^2$

4\. D: Yeah,except for somehow--I mean I don't really know--but I doubt that's the triangle of minimum area--well, o.k. we'll try it.

5\. K: Largest area. Well, it is the only way we can figure out the area.

6\. D: All right.

7\. K: But for an isosceles we can do almost the same thing. This is 1/2(A). So that we know that the area is $(A/2)\sqrt{C^2-(A/2)^2}$. The perimeter = A + B + C and the height equals $\sqrt{C^2-(A/2)^2}$.



$b=c$

8\. D: All right.

9\. K: Now what do we do. We've got to figure out the largest area.

10\. D: Isn't it the minimum?

11\. K: The largest area.

$A=\frac{1}{2}(a)\sqrt{c^2-(\frac{1}{2}a)^2}$

$a+b+c=P$

$h=\sqrt{c^2-(\frac{a}{2})^2}$

12\. D: So actually if we can get A--we have to get everything in terms of one variable and take the derivative, right? Basically?

13\. K: Yeah,well--

14\. D: Well, I still don't know if we should do--I mean we can find an area for this and can find an area for that, granted, but if we

ever come to a problem like this--I mean we don't know--we have no idea as of yet with a given perimeter what's going to be that.

15. K: Right.

16. D: So, there--I mean--you can do that again but then what do you do?

17. K: Then we're stuck, right? Usually, you know, you could probably take a guess as to what kind of triangle it would be--like you could say it is a right triangle or an isosceles--I think it is an equilateral, but I don't know how to prove it.

18. D: Umma.

19. K: So we have to figure out some way to try to prove that.

20. D: All right, a good guess is that it is an equilateral, then why don't we try an isosceles and if we can find that these two sides have to be equal to form the maximum area, then we can find that--then we should be able to prove that side also has to be equal.

21. K: O.k. so B will be equal to C, so the perimeter P = A + 2B, or A + 2C = P.

22. D: All right.

23. K: Ummmm.

24. D: See what we've got.

$1/2\sqrt{c^2-1/4}$

25. K: Fix A as a constant then we can do this, solve that for C.

$1/2(c^2-1/4)^{1/2}=0$

26. D: All right.

$1/4(c^2-1/4)^{-1/2}(2c)=0$

27. K: For a maximum area we've got 1/2, let's say A = 1, $\sqrt{C^2 -1/4}$, right? Maximum area: $1/2(C^2-1/4)^{1/2} = 0$.

$2c=0$

$c=0$

28. D: C^2 minus what?

29. K: $(1/2)^2$, yeah, $(1/2)^2$. A/2, where A = 1. O.k.?

30. D: Ah, ah.

31. K: Mumbling--this is $1/4(C^2-1/4)^{-1/2}$. 2C, so we know that 2C has to = 0 and C = 0 and we are stuck!

32. D: We should have taken a derivative in it and everything, you think?

33. K: Yeah, that's the derivative of that. So does it help us? My calculus doesn't seem to work anymore.

34. D: The thing is--pause--you are letting C be the variable, holding A constant. So what was your formula--1/2 base times square root.

35. K: The base A times the square root times the height which is a right triangle to an isosceles which is --so it is $C^2-(A/2)^2$ which would give you this height.

36. D: $A^{2/4}$, no, $A^{2/2}$, no, $(A/2)^2$.

37. K: How about P = , ... no, C = P -A/2? Should we try that--

38. D: No, see part of the thing is, I think that for here we're just saying we have a triangle, an isosceles triangle, what is going to be the largest area? Largest area.

39. K: Largest area--set its derivative equal to 0.

40. D: All right. Well the largest area or the smallest area--I mean--if we are going to take a derivative--I mean--what's going to happen is you have a base and it's going to go down like that--I mean--we don't set any conditions--we're leaving P out of that.

41. K: Ah, ah.

42. D: That's absolutely what we have to stick in.

43. K: We've got C and a P-A over 2.

44. D: P -A over 2.

45. K: Formula--isosceles.

46. D: A + 2B = P--all right?

47. K: Shall we try that--mumbling. -A over 2--we've got to have a minus 1/4 PA--

$\frac{A}{2}\left(\left(\frac{P-a}{2}\right)^2-\frac{a^2}{4}\right)^{1/2}$

48. D: Well, then you can put A back in--then you can have everything in terms of A, right? Using this formula, we have the area and we have a--

$\frac{a}{2}\left(\frac{P^2-2a+a^2}{4}-\frac{a^2}{4}\right)$

49. K: All right--P--so that's $A/2\left(\frac{P^2-2A+A^2}{4}-\frac{A^2}{4}\right)^{1/2}$ and that's $A/2\left(\frac{P^2-2A}{4}\right)^{1/2}$...(mumbling and figuring)

$\frac{a}{2}\left(\frac{P^2-2a}{4}\right)^{-1/2}(2p-2)$

50. D: Wait a minute--you just took the derivative of this right here?

51. K: This times the derivative of this plus this times the derivative of this.

$\left(\frac{P^2-2a}{4}\right)^{1/2}(1/2)=$

52. D: Oh.

53. K: Mumbling and figuring...$A/4\left(\frac{P^2-2A}{4}\right)^{-1/2}(2P-2)+\left(\frac{P^2-2A}{4}\right)^{1/2}$ $1/2=0$...so $\frac{2AP-2A}{4}+\frac{P^2-2A}{8}=0$.

$\frac{2ap-2a}{4}+\frac{P^2-2a}{8}=0$

54. D: So can we get A in terms of P?

55. K: P^2--

8
P^2

56. D: $8P^2 - 8P^2$ bring the P^2 on this side and multiply it by 8 and we'll have a qudratic in terms--no we won't--then we can just have A we can factor out in the equation--you see.

57. K: O.k. $P^2 =$

58. D: $-8P^2$--oh, are we going to bring everything else to the other side?

59. K: Yeah, 2A- +4A - 4AP x 8--No--

60. D: That's not right. Well, the 8 we can just multiply--

61. K: P^2 = all this.

62. D: Right.

63. K: $P^2 - 4AP =$ --this isn't getting us anywhere.

$p^2 = 2a + 4a + 4ap$

64. D: $P^2 =$ factor out the A--then we can get A in terms of P.

65. K: $P^2 = 2A$--so you've got $A = \frac{P^2}{6+4P}$ --

$p^2 = 2a(3+2p)$

66. D: So if we have an isosceles triangle and A has = to--

67. K: be equal to that--

$\frac{p^2}{3+2p} = 2a$

68. D: And if A has to be equal to that and B and C are equal--

69. K: So, B = --(whistles)

70. D: B = P- that.

$a = \frac{p^2}{6+4p}$

71. K: 2B = P-A over 2.

72. D: No we aren't getting anything here--we're just getting--thing is that we assumed B to be equal to C so of course, I mean--that doesn't--we want to find out if B is going to be equal to C and we have a certain base--let's start all over, and forget about this. All right, another triangle. Certain altitude.

73. K: Well, let's try to assume that it is an equilateral.

74. D: All right.

75. K: Sides--mumbling--perimeter equals 3S, right?

76. D: Yeah, but wait a minute--that's still not going to really help us--what are we going to do--simply assume that it is an equilateral. We're just going to get that it is an equilateral, of course it is going to be an equilateral if we assume that.

77. K: True.

78. D: We want to prove that it is an equilateral if we think it is. If we want to do anything we can--

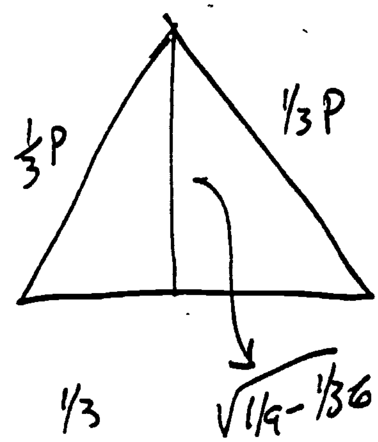
79. K: Yeah, how do you prove it?

80. D: Well, we can make up a perimeter--we don't need a perimeter P, do we? So,--

81. K: Where are you going to get area formula in the form of P?

82. D: We want to maximize the area so that we can prove--o.k. we have the given base--we'll set our base equal to something.

83. K: Yeah, mumbling, P, or something--I don't know.



84. D: Then the other two sides have to add up to P.

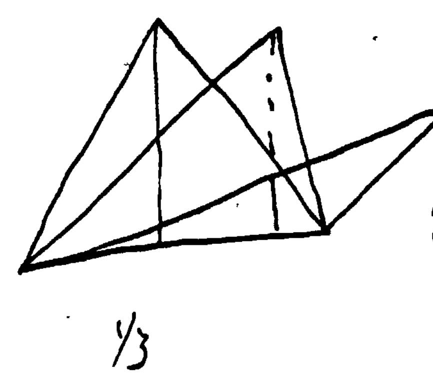
85. K: We--how about we say--let's start with an equilateral, just for the hell of it--see what happens. You get 1/3P, 1/3P and 1/3P. And this is 1/9 - 1/36 which is the height--

86. D: Now the thing we want to do is say--o.k. if we shorten this side at all and then what's going to happen to the height--if we leave this the same.

87. K: We can't shorten it.

88. D: And we shorten this side--sure we can--

89. K: Well--



90. D: We can have a--this equal to 1/3 and then a--this equal to--well you're going to have--I mean--

91. K: Aha.

92. D: This is going to get longer like that. Now we can see from this that all that is going to happen is that the base is going to get shorter so we know from that as far as leaving the base constant goes if we move--if we shorten this side then it is going to--somehow the point's going to go down in either direction.

93. K: Semicircle.

94. D: Right. That proves that we have to have an equilateral.

95. K: No, it proves an isosceles.

96. D: No, isosceles, I mean. All right from that if we set--we know that those two have to be equal so if we set this base equal to anything--it doesn't have to be 1/3P--we can also show that if this goes down--the area is going to get smaller, so the constant base then the height is going to get shorter and shorter and is getting smaller and smaller actually.

97. K: O.k., o.k.

98. D: In this case if it goes down to this side, we're going to have again a smaller angle here, shorter base here--and [noise].

99. K: So we get--so we know it is an equilateral--well prove it.

100. D: I don't know that's not a rigorous proof, but it is a proof--good enough for me.

101. K: Proves that an equilateral has the largest area.

102. D: Oh, we're talking about the largest area.

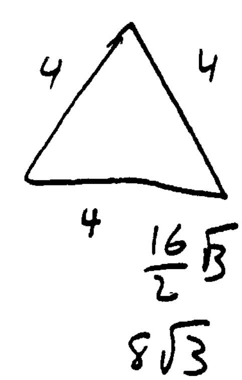
103. K: Yeah.

104. D: Oh, we just did.

105. K: We have to prove it has fixed number P--perimeter.

106. D: Well we already--we assumed that we have a fixed P, all right? I mean this is a proof as far as I.

107. K: Well, we've shown that an equilateral has the largest area. We haven't shown that if you have a certain set perimeter, let's say a right triangle, with a perimeter which is the same--we will not have a larger area.



108. D: No, but we have because we have shown with the set perimeter--o.k. we know that--

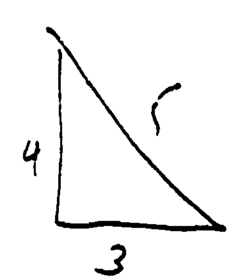
109. K: Well what if we have 3, 4, 5 with an equilateral being 4, 4, 4--

110. D: 3 4, 5 is what? Mumbling.

111. K: 12. So this area will be 6 and this area will be side squared 16. --o.k. that will have the largest area.

112. D: What's that 1.7?

113. K: Yeah, 8 is still greater than 6 and that's greater than 1.



114. D: Oh, yeah, that's right. Yeah, but the thing is if we have a fixed dimension, we already showed that, o.k. what is going to happen is as this

side gets longer--say we use 4 as a base here, so then what's going to happen--well say we use 3 as a base, just so we won't have an equilateral when we are done--what's going to happen as 4 gets longer and 5 gets shorter--it's going to go upwards. The optimum area--the maximum area is going to be right there. Because you've got--

115. K: Right.

116. D: This angle and that height. If you make this angle any less--maybe let me draw a picture--

117. K: I can understand that--this will give us largest area, but how can we prove this bottom is one-quarter--1/3 the area of the perimeter?

118. D: Well, remember all the problems we've done where we say--o.k. let me just start from here once more--so that we have 3, 4, 5--is that what you have--because that's going to be 5. Wasn't a very good 3, 4, 5 anyway. So you start out with 3, 4, 5--all right, we pick the 3 has the base, right?

119. K: Aha.

120. D: All right, it's 5--mumbling--if we have 3 as the base--and this is a little bit off an isosceles, but if we draw an isosceles as 3 as the base--o.k. we've got a right angle--that's got to be the maximum--mumbling--(height?) because if it goes any--

121. K: Right.

122. D: Over this way, it is going to go down.

123. K: O.k.

124. D: All right, so remember the argument we've used--well if we--

125. K: Yeah, I can show that, but what you're not showing is--what you're not proving is that--

126. D: That it has to be an equilateral?

127. K: Right. But you're not showing that this side is 1/3 the perimeter.

128. D: Right. I'm showing--first of all it has to be an isosceles. Right.

129. K: Right.

130. D: It has to be an isosceles--that means that we've got these three sides and those two are equal--right?

131. K: Umma.

132. D: Right--so now I pick this side as my base--I already picked--if that side is my base then the maximum area would have to have an isosceles--so I turn around--this side is my--

133. K: That I understand as proof, but you're not showing me that this is 1/3 the perimeter--mumbling.

134. D: If we have an isosceles triangle--if we have an equilateral triangle--then each side has to be 1/3 the perimeter--that's the whole thing about an equilateral triangle.

135. K: I know--o.k.

136. D: First we know it must be an isosceles, right?

137. K: Umma.

138. D: O.k.

139. K: I understand this.

140. D: If it is an isosceles, it must be an equilateral, right?

141. K: All right.

142. D: And if it must be an equilateral--all three sides must be equal and if the perimeter is P, all three sides must be 1/3P.

143. K: O.k. I've got it.